OCR
RECOGNISING ACHIEVEMENT

Wednesday 30 January 2013 – Morning

A2 GCE MATHEMATICS

4727/01 Further Pure Mathematics 3

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4727
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

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1 Two planes have equations

$$x + 2y + 5z = 12 \quad \text{and} \quad 2x - y + 3z = 5.$$

(i) Find the acute angle between the planes. **[3]**

(ii) Find a vector equation of the line of intersection of the planes. **[4]**

2 The elements of a group G are the complex numbers $a + b\text{i}$ where $a, b \in \{0, 1, 2, 3, 4\}$. These elements are combined under the operation of addition modulo 5.

(i) State the identity element and the order of G. **[2]**

(ii) Write down the inverse of $2 + 4\text{i}$. **[1]**

(iii) Show that every non-zero element of G has order 5. **[3]**

3 Solve the differential equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y = x^4\mathrm{e}^{2x}$ for y in terms of x, given that $y = 0$ when $x = 1$. **[8]**

4 The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$$

respectively.

(i) Find the shortest distance between the lines. **[5]**

(ii) Find a cartesian equation of the plane which contains l_1 and which is parallel to l_2. **[2]**

5 **(i)** Solve the equation $z^5 = 1$, giving your answers in polar form. **[2]**

(ii) Hence, by considering the equation $(z + 1)^5 = z^5$, show that the roots of

$$5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$$

can be expressed in the form $\dfrac{1}{\mathrm{e}^{\mathrm{i}\theta} - 1}$, stating the values of θ. **[5]**

6 The differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = \sin kx$ is to be solved, where k is a constant.

(i) In the case $k = 2$, by using a particular integral of the form $ax\cos 2x + bx\sin 2x$, find the general solution. **[7]**

(ii) Describe briefly the behaviour of y when $x \to \infty$. **[2]**

(iii) In the case $k \neq 2$, explain whether y would exhibit the same behaviour as in part **(ii)** when $x \to \infty$. **[2]**

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7 Let $S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \ldots + e^{10i\theta}$.

(i) (a) Show that, for $\theta \neq 2n\pi$, where n is an integer,

$$S = \frac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}.$$ **[4]**

(b) State the value of S for $\theta = 2n\pi$, where n is an integer. **[1]**

(ii) Hence show that, for $\theta \neq 2n\pi$, where n is an integer,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = \frac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)} - \frac{1}{2}.$$ **[3]**

(iii) Hence show that $\theta = \frac{1}{11}\pi$ is a root of $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 10\theta = 0$ and find another root in the interval $0 < \theta < \frac{1}{4}\pi$. **[4]**

8 A multiplicative group H has the elements $\{e, a, a^2, a^3, w, aw, a^2w, a^3w\}$ where e is the identity, elements a and w have orders 4 and 2 respectively and $wa = a^3w$.

(i) Show that $wa^2 = a^2w$ and also that $wa^3 = aw$. **[6]**

(ii) Hence show that each of aw, a^2w and a^3w has order 2. **[4]**

(iii) Find two non-cyclic subgroups of H of order 4, and show that they are not cyclic. **[4]**

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Question			Answer	Marks	Guidance	
1	(i)		$\cos\theta = \dfrac{\left\|\begin{pmatrix}1\\2\\5\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\3\end{pmatrix}\right\|}{\sqrt{1^2+2^2+5^2}\sqrt{2^2+(-1)^2+3^2}} = \dfrac{15}{\sqrt{30}\sqrt{14}}$ $\theta = 0.750$ or $43.0°$	M1 A1 A1 **[3]**	Accept unsimplified If zero, then **sc1** for $n_1.n_2 = 15$ seen	
1	(ii)		$\begin{pmatrix}1\\2\\5\end{pmatrix}\times\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}11\\7\\-5\end{pmatrix}$ (eg) $x=0 \Rightarrow 2y+5z=12, -y+3z=5 \Rightarrow y=1, z=2$ $\mathbf{r}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}11\\7\\-5\end{pmatrix}$ **Alternative:** Find one point Find a second point and vector between points multiple of $\begin{pmatrix}11\\7\\-5\end{pmatrix}$ $\mathbf{r}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}11\\7\\-5\end{pmatrix}$	M1 A1 M1 A1 **[4]** M1 M1 A1 A1	oe vector form	M1 requires evidence of method for cross product or at least 2 correct values calculated or any valid point e.g.(-11/7, 0, 19/7) (22/5, 19/5, 0) Must have full equation including '**r** ='

Question			Answer	Marks	Guidance	
			Alternative: Solve simultaneously Point found Direction found $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 7 \\ -5 \end{pmatrix}$	M1 A1 A1 A1	to at least expressions for x,y,z parametrically, or two relationship between 2 variables.	
2	(i)		identity 0 + 0i order 25	B1 B1 **[2]**	Or '0'	
2	(ii)		$3+\mathrm{i}$	B1 **[1]**		
2	(iii)		$5(a+b\mathrm{i}) = 5a+5b\mathrm{i} = 0+0\mathrm{i}$ every non-zero element has order 5 or 25 So order is 5	M1 M1 A1 **[3]**	Shows 5 times any element equals e Attempt to show that order ≠ 2,3,4 Argument is convincing, exhaustive and conclusive.	Must consider all(non-zero) elements
3			$\frac{\mathrm{d}y}{\mathrm{d}x} - 3\frac{y}{x} = x^3\,\mathrm{e}^{2x}$ $I = \exp\left(\int -\frac{3}{x}\mathrm{d}x\right) = \mathrm{e}^{-3\ln x}$ $= x^{-3}$ $x^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} - 3x^{-4}y = \mathrm{e}^{2x}$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-3}y\right) = \mathrm{e}^{2x}$ $x^{-3}y = \frac{1}{2}\mathrm{e}^{2x} + A$ $x=1, y=0 \Rightarrow A = -\frac{1}{2}\mathrm{e}^2$ $y = \frac{1}{2}x^3\left(\mathrm{e}^{2x} - \mathrm{e}^2\right)$	M1 M1 A1 M1 M1 A1 M1 A1 **[8]**	Divide by x Multiply and recognise derivative Integrate Use condition	

Question			Answer	Marks	Guidance	
4	(i)		$\begin{pmatrix}2\\3\\-1\end{pmatrix}\times\begin{pmatrix}4\\-1\\-1\end{pmatrix}=\begin{pmatrix}-4\\-2\\-14\end{pmatrix}=-2\begin{pmatrix}2\\1\\7\end{pmatrix}$	M1 A1	Or any multiple	
			$\begin{pmatrix}3\\0\\1\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}2\\-2\\0\end{pmatrix}$	B1	Or negative	Or use of n.(a_1 + pb_1 +kn) = n.(a_2 + qb_2) **B1** followed by attempt to calculate magnitude of kn **M1**
			shortest distance $=\dfrac{\left\lvert\begin{pmatrix}2\\-2\\0\end{pmatrix}\bullet\begin{pmatrix}2\\1\\7\end{pmatrix}\right\rvert}{\sqrt{2^2+1^2+7^2}}=\dfrac{2}{\sqrt{54}}$ oe	M1 A1 **[5]**	Component of their vector in their direction	
4	(ii)		$2x+y+7z=\ldots$ $\ldots 11$	B1ft B1 dep **[2]**	ft from 4(i) only if 1st M1 mark gained If zero, then **sc 1** for any correct **vector** equation.	
5	(i)		$1, e^{\frac{2}{5}\pi i}, e^{\frac{4}{5}\pi i}, e^{\frac{6}{5}\pi i}, e^{\frac{8}{5}\pi i}$ oe polar form	M1 A1 **[2]**	Attempt roots	e.g. gives roots in an incorrect form.

Question			Answer	Marks	Guidance	
5	(ii)		$z^5 = (z+1)^5 = z^5 + 5z^4 + 10z^3 + 10z^2 + 5z + 1$	M1		
			$\Leftrightarrow 5z^4 + 10z^3 + 10z^2 + 5z + 1 = 0$	A1		
			so $z + 1 = z\,\mathrm{e}^{\frac{2k}{5}\pi \mathrm{i}}$, $k = 0,1,2,3,4$	M1		
			$k = 0$ no solution	B1	soi	
			$1 = z\left(\mathrm{e}^{\frac{2k}{5}\pi \mathrm{i}} - 1\right)$			
			$z = \dfrac{1}{\mathrm{e}^{\frac{2k}{5}\pi \mathrm{i}} - 1}$, $k = 1,2,3,4$	A1	If B0, then give A1 ft for correct solution plus $k = 0$	
				[5]		
6	(i)		PI: $y = ax\cos 2x + bx\sin 2x$			
			$\dfrac{\mathrm{d}y}{\mathrm{d}x} = a\cos 2x - 2ax\sin 2x + b\sin 2x + 2bx\cos 2x$	B1	For correct $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ or better	
			$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -4a\sin 2x - 4ax\cos 2x + 4b\cos 2x - 4bx\sin 2x$			
			in DE: $-4a\sin 2x - 4ax\cos 2x + 4b\cos 2x - 4bx\sin 2x$ $+4(ax\cos 2x + bx\sin 2x)$	M1	Differentiate twice and substitute	
			compare coefficients: $-4a = 1,\ 4b = 0$	M1		
			$\Rightarrow a = -\frac{1}{4}, b = 0$	A1		
			AE: $\lambda^2 + 4 = 0$	M1	For correct auxiliary equation and attempt to solve	
			$\lambda = \pm 2\mathrm{i}$			
			CF: $A\cos 2x + B\sin 2x$	A1	oe form	
			GS: $y = \left(A - \frac{1}{4}x\right)\cos 2x + B\sin 2x$	A1ft	Must be real and contain 2 unknowns	
				[7]		

Question			Answer	Marks	Guidance	
6	(ii)		oscillations unbounded	B1 B1 **[2]**	oe (accept sketch) dep consistent with 6(i) oe (accept sketch) dep consistent with 6(i) If zero, then **sc1** for recognition that xcos2x term becomes dominant	
6	(iii)		If $k \neq 2$ then PI $y = \alpha \cos kx + \beta \sin kx$ So bounded oscillations	B1 B1 **[2]**	 oe (accept sketch)	
7	(i)	(a)	$e^{i\theta} + e^{2i\theta} + \ldots + e^{10i\theta} = \dfrac{e^{i\theta}\left(\left(e^{i\theta}\right)^{10} - 1\right)}{e^{i\theta} - 1}$ $= \dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}}$ $= \dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta} - 1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}$	M1 A1 M1 A1 **[4]**	Sum of a GP AG	
7	(i)	(b)	$\theta = 2n\pi \Rightarrow \text{sum} = 10$	B1 **[1]**		

Question			Answer	Marks	Guidance	
7	(ii)		$\cos\theta+\cos 2\theta+\ldots+\cos 10\theta=\mathrm{Re}\left(\dfrac{e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)}{2i\sin\left(\frac{1}{2}\theta\right)}\right)$	M1	Take real parts	
			$=\dfrac{\mathrm{Re}\left(-i\,e^{\frac{1}{2}i\theta}\left(e^{10i\theta}-1\right)\right)}{2\sin\left(\frac{1}{2}\theta\right)}=\dfrac{\mathrm{Re}\left(-i\,e^{\frac{21}{2}i\theta}+i\,e^{\frac{1}{2}i\theta}\right)}{2\sin\left(\frac{1}{2}\theta\right)}$	M1	Manipulate expression	Must at least make genuine progress in sorting real part of numerator, or in converting numerator to trig terms.
			$=\dfrac{\sin\left(\frac{21}{2}\theta\right)-\sin\left(\frac{1}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}$ $=\dfrac{\sin\left(\frac{21}{2}\theta\right)}{2\sin\left(\frac{1}{2}\theta\right)}-\dfrac{1}{2}$	A1 **[3]**	AG	
7	(iii)		$\cos\frac{1}{11}\pi+\cos\frac{2}{11}\pi+\ldots+\cos\frac{10}{11}\pi=\dfrac{\sin\left(\frac{21}{22}\pi\right)}{2\sin\left(\frac{1}{22}\pi\right)}-\dfrac{1}{2}$	M1		For second M1, must convince that solution is exact and not simply from calculator.
			But $\sin\frac{21}{22}\pi=\sin\left(\pi-\frac{21}{22}\pi\right)=\sin\frac{1}{22}\pi$	M1		
			So RHS $=\frac{1}{2}-\frac{1}{2}=0$, so $\frac{1}{11}\pi$ is a root	A1	AG	
			Using $\sin(2\pi+x)=\sin x$ gives $2\pi+\frac{1}{2}\theta=\frac{21}{2}\theta\Rightarrow\theta=\frac{1}{5}\pi$	A1 **[4]**		

8	(i)		$wa^2 = waa = a^3wa = a^3a^3w$ $= a^4a^2w = ea^2w$ $= a^2w$ Either result $\Rightarrow wa^3 = a^3wa^2$ $= a^3a^2w$ $= eaw = aw$	M1 B1 A1 M1 M1 A1 **[6]**	Use $wa = a^3w$ to simplify Use $a^4 = e$ (oe) in either proof Complete argument AG AG	
8	(ii)		$(aw)^2 = (aw)(aw)$ $= awwa^3 = aea^3 = a^4 = e$ so order 2 $(a^2w)(a^2w) = a^2wwa^2 = a^2ea^2 = a^4 = e$ so order 2 $(a^3w)(a^3w) = a^3wwa = a^3ea = a^4 = e$ so order 2	M1 M1 A1 A1 **[4]**	for squaring any of elements for attempt to simplify to e for at least two squared elements shown equal to e for complete argument	
8	(iii)		$\{e, a^2, w, a^2w\}$ $\{e, a^2, aw, a^3w\}$ a^2, w, aw, a^2w, a^3w all of order 2 so not cyclic as no element of order 4 in either	B1 B1 M1 A1 **[4]**	 Consider orders Or considers form {e, x, y, xy} where x, y order 2 Dep on both groups correct	Condone equivalents Condone 'no generator' or 'Klein (V) group' in place of 'no element of order 4'